数理解析研究所講究録
第 1718 巻 2010 年 41-51

AMALGAMATION, THE BOUNDARY PROPERTY, AND HOMOLOGY

BYUNGHAN KIM

In this note we report recent research works by the author, John Goodrick, and Alexei Kolesnikov which will come out as two papers [3][4]. As well-known when the intensive studies on simplicity theory started in mid 90s, the author and Pillay [8] proved that any simple theory satisfies 'the Independence Theorem', which nowadays is called 'type-amalgamation' or '3-amalgamation'. Then already it was noticed that typical simple algebraic structures such as a vector space over a finite field with a bilinear map need not satisfy higher dimensional amalgamation.

When research direction moved towards *geometric* simplicity theory in early 2000, in particular trying to generalize the group configuration theorem to the simple theory context, surprisingly it was revealed that higher dimensional amalgamation is to do with this problem. The author together with de Piro and Millar showed that under 4-amalgamation, given a hyperdefinable group configuration, the hyperdefinable homogeneous space equivalent to the given configuration can be obtained [1][6]. Concurrently the notion of n-amalgamation is studied by A. Kolesnikov [9] and then together with the author and Tsuboi [7]. It is Hrushovski who then pointed out that the failure of 4-amalgamation in a stable theory is to do with a certain definability of a groupoid [5]. In a similar manner he introduced more generalized notion of imaginaries obtained from finitary definable groupiods. (The usual imaginaries are those come from trivial groupoids.) Elimination of such should be involved to establish n-amalgamation even in a stable theory. As a generalization of an eq-construction for elimination of ususal imaginaries, a *stably extendible* extension can be constructed so that we can freely assume n-amalgamation for stable theories. Whether such construction for simple theories is possible remains as an open question.

Goodrick and Kolesnikov then studied more on the relationship between definable groupoids and the failure of 4-amalgamation in stable theories [2]. Using ingenious 'groupoid' configuration argument they

---

The author was supported by NRF grant 2009-0075931.

explicitly constructed a non-eliminable finitary groupoid in a stable structure where 4-amalgamation does not hold. In [3], we extend the results of [2] in several directions. We show that the mentioned groupoid in a stable theory without 4-amalgamation should be abelian. We also try to generalized the results in the context of simple theories. More importantly we single out and studied the notion of the boundary property used in [5][1]. Using the notion we succeed to answer the question in [5] positively. Namely under an induction hypothesis, the equivalence of n-uniqueness and $(n+1)$-amalgamation is established.

In [4], we begin to introduce homology theory for simple theories which is to do with the issues and notions so far mentioned. Generalized amalgamation is to do with the computation of homology groups. Typical examples having non-trivial homology groups come from the groupoids examples, and their binding groups are equal to the homology groups.

Throughout this note T is simple. We assume T has elimination of hyperimaginaries. But this assumption is mainly for terminological simplicity, can be removable by working in $\mathcal{M}^{heq}$ instead of $\mathcal{M}^{eq} \models T$.

1. The amalgamation properties

Definition 1.1. Recall that by a *category* $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$, we mean a class $\mathrm{Ob}(\mathcal{C})$ of members called *objects* of the category; equipped with a class $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{Mor}(a,b) |\ a, b \in \mathrm{Ob}(\mathcal{C})\}$ where $\mathrm{Mor}(a,b) = \mathrm{Mor}_{\mathcal{C}}(a,b)$ is the class of *morphisms* between objects a, b (we write $f : a \to b$ to denote $f \in \mathrm{Mor}(a,b)$); and composition maps $\circ : \mathrm{Mor}(a,b) \times \mathrm{Mor}(b,c) \to \mathrm{Mor}(a,c)$ for each $a, b, c \in \mathrm{Ob}(\mathcal{C})$ such that

(1) (Associativity) if $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$ holds, and
(2) (Identity) for each object c, there exists a morphism $1_c : c \to c$ called the identity morphism for c, such that for $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.

Note that any ordered set $(P, \leq)$ is a category where objects are members of P, and $\mathrm{Mor}(a,b) = \{(a,b)\}$ if $a \leq b$; $= \emptyset$ otherwise.

Now we recall a functor F between two categories $\mathcal{C}, \mathcal{D}$.

Definition 1.2. The *functor* F sends an object $c \in \mathrm{Ob}(\mathcal{C})$ to $F(c) \in \mathrm{Ob}(\mathcal{D})$; and a morphism $f \in \mathrm{Mor}_{\mathcal{C}}(a,b)$ to $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(a), F(b))$ in such a way that

(1) (Associativity) $F(g \circ f) = F(g) \circ F(f)$ for $f : a \to b$, $g : b \to c$;
(2) (Identity) $F(1_c) = 1_{F(c)}$.

Definition 1.3. Let $\mathcal{C}^* = \mathcal{C}^*(T)$ be the category of algebraically closed substructures with elementary embeddings.

(1) A functor $f : \mathcal{P}^-(n) \to \mathcal{C}^*$ is given, where $\mathcal{P}^-(n) = \mathcal{P}(n) - \{n\}$. For $u \subseteq w \in \mathcal{P}^-(n)$, $f^u_w(u)$ is an image of $f(u)$ in $f(w)$ under $f((u, w))$. Now f is said to be *independence preserving (i.p.)* if for $u \in \mathcal{P}^-(n)$,
 (a) $\{f^{\{i\}}_u(\{i\}) | i \in u\}$ is $f^{\emptyset}_u(\emptyset)$-independent;
 (b) $f(u) = \mathrm{acl}(\bigcup\{f^{\{i\}}_u(\{i\})|\ i \in u\})$.

(2) We say T has *n-amalgamation* (equivalently *n-existence*) if any i.p. functor $f : \mathcal{P}^-(n) \to \mathcal{C}^*$ can be extended to an i.p. $\hat{f} : \mathcal{P}(n) \to \mathcal{C}^*$.

(3) We say T has *n-uniqueness* if in (2), any two completions $\hat{f}, \hat{f}'$ of f are isomorphic.

Fact 1.4. *(K., Pillay) Any T has 3-amalgamation.*

But there are examples having n-amalgamation, but not $(n + 1)$-amalgamation, for each $n \geq 3$.

Theorem 1.5. *(The group configuration theorem) Assume T has 4-amalgamation. Then given a type-definable group configuration, there is the type-definable homogeneous space which is equivalent to the configuration.*

Corollary 1.6. *(K., dePiro, Millar) In any non-trivial modular theory with 4-amalgamation, an infinite vector space over a ring of endomorphisms whose linear independence coincides with forking independence is type-definable.*

Does n-amalgamation hold in all stable theories?

No. n-amalgamation holds only over models. But in general over an algebraically closed set, even 4-amalgamation need not hold in stable theories.

Example 1.7. Consider $[I]^2 = \{\{a, b\}|a \neq b \in I\}$ where I infinite. Let $B = [I]^2 \times \{0, 1\}$ where $\{0, 1\} = \mathbb{Z}_2$.

Also let $E \subseteq I \times [I]^2$ be a membership relation, and let P be a subset of B^3 such that $((w_1, \delta_1)(w_2, \delta_2)(w_3, \delta_3)) \in P$ iff there are distinct $a_1, a_2, a_3 \in I$ such that for $\{i, j, k\} = \{1, 2, 3\}$, $w_i = \{a_j, a_k\}$ (i.e. w_1, w_2, w_3 are edges of a triangle), and $\delta_1 + \delta_2 + \delta_3 = 0$.

Let $M_0 = (I, [I]^2, B;\ E, P;\ \mathrm{Pr}_1 : B \to [I]^2)$. Then M_0 is stable.

$T_0 = Th(M_0)$ does not have 4-amalgamation over $\emptyset = \mathrm{acl}(\emptyset)$.

Note first that $\mathrm{dcl}(\emptyset) = \mathrm{acl}(\emptyset)$, and for $a \in I$, $\mathrm{dcl}(a) = \mathrm{acl}(a)$. Now choose distinct $a_1, a_2, a_3, a_4 \in I$. For $\{i, j, k\} \subseteq \{1, 2, 3, 4\}$, fix an enumeration $\overline{a_ia_j} = (b_{ij}, ...)$ of $\mathrm{acl}(a_ia_j)$ where $b_{ij} = (\{a_i, a_j\}, \delta) \in B = [I]^2 \times \{0, 1\}$. Let $r_{ij}(x_{ij}) = \mathrm{tp}(\overline{a_ia_j})$, and let x^1_{ij} be the variable for b_{ij}. Note that $b_{ij} = (\{a_i, a_j\}, \delta)$ and $b'_{ij} = (\{a_i, a_j\}, \delta + 1)$ have the same type over a_ia_j. Hence there is $(\overline{a_ia_j})' = (b'_{ij}, ...)$ also realizing $r_{ij}(x_{ij})$. Therefore we have complete types $r_{ijk}(x_{ijk})$, $r'_{ijk}(x'_{ijk})$ both extending $r_{ij}(x_{ij}) \cup r_{ik}(x_{ik}) \cup r_{jk}(x_{jk})$ realized by some enumerations of $\mathrm{acl}(a_ia_ja_k)$ such that $P(x^1_{ij}x^1_{ik}x^1_{jk}) \in r_{ijk}$ whereas $\neg P(x^1_{ij}x^1_{ik}x^1_{jk}) \in r'_{ijk}$. Then it is easy to see that $r_{123} \cup r_{124} \cup r_{134} \cup r'_{234}$ is inconsistent.

Rather surprisingly, for a stable theory, having 4-amalgamation is to do with (non) definability of a certain groupoid; and a certain definable closure relation which we call the boundary property.

Definition 1.8. A category $\mathcal{G} = (\mathrm{Ob}(\mathcal{G}), \mathrm{Mor}(\mathcal{G}))$ is called a *groupoid* if any morphism $f : a \to b$ has an inverse $f^{-1} : b \to a$ so that $f \circ f^{-1} = 1_b$ and $f^{-1} \circ f = 1_a$.

Any $\mathrm{Mor}(a, a)$ with the composition map forms a group, and if $\mathrm{Mor}(a, b)$ is non-empty, then $\mathrm{Mor}(a, a)$ and $\mathrm{Mor}(b, b)$ are isomorphic as groups.

Definition 1.9.
(1) A groupoid $\mathcal{G} = (\mathrm{Ob}(\mathcal{G}), \mathrm{Mor}(\mathcal{G}))$ is said to be *(type-)definable* if if both $\mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}(\mathcal{G})$ are (type-)definable, as well as the composition operation and the domain, range and identity maps.
(2) A type-definable groupoid $\mathcal{G}$ is said to be *eliminable* if there are a definable groupoid $\{*\}$ with a single object and fully faithful definable functor $F : \mathcal{G} \to \{*\}$.

Fact 1.10. *(Hrushovski; Goodrick, Kolesnikov) Let T be stable. The following are equivalent*

(1) *T has 3-uniqueness.*
(2) *T has 4-amalgamation.*
(3) *T has $B(3)$. Namely, for any A-independent set $\{a_1, a_2, a_3\}$ with $A \subseteq a_i = \mathrm{acl}(a_i)$,*

$$\mathrm{dcl}(\mathrm{acl}(a_3a_1)\,\mathrm{acl}(a_3a_2)) \cap \mathrm{acl}(a_2a_1) = \mathrm{dcl}(a_1a_2).$$

(4) *Any connected finitary definable groupoid is eliminable.*

In sprit of previous Fact (4), one may ask whether we can safely assume 4-amalgamation for any stable theory T after extending $\mathcal{M}(\models T)$ to a certain larger structure in which $\mathcal{M}$ is *stably embedded.*

Hrushovski answered this positively. He showed that for such stable $\mathcal{M}$ there is $\mathcal{M}^* \models T^*$ in $\mathcal{L}^*(\supseteq \mathcal{L})$ such that $\mathcal{M}$ is *stably embedded* into $\mathcal{M}^*$, and $\mathcal{M}^*$ has n-amalgamation for all n.

We may write $\mathcal{M}^*$ as $\mathcal{M}^{geq}$.

In short, like $\mathcal{M}^{eq}$, wlog, we can assume $\mathcal{M} = \mathcal{M}^{geq}$ when T is stable.

Whether we can extend the result to the context of unstable simple theories is an open question.

Definition 1.11. In a simple theory, $B(n)$ is the following property: for any A-independent set $\{a_1, \ldots, a_n\}$ with $A \subseteq a_i$,

$$\begin{aligned}\mathrm{dcl}(\overline{a}_{\hat{1},\ldots,n}, \ldots, \overline{a}_{1,\ldots\widehat{n-1},n}) \cap \overline{a}_{1,\ldots,n-1} \\ = \mathrm{dcl}(\overline{a}_{\hat{1},\ldots,n-1}, \ldots, \overline{a}_{1,\ldots\widehat{n-1}}),\end{aligned}$$

where $\overline{a}_{\hat{1},\ldots,n}$ denotes $\mathrm{acl}(a_2, \ldots, a_n)$ and so on.

Fact 1.12. *T simple. TFAE.*

(1) *T has $B(n)$.*

(2) *For any A-independent set $\{a_1, \ldots, a_n\}$ with $A \subseteq a_i$,*

$$\mathrm{tp}(\overline{a}_{1,\ldots,n-1}/\overline{a}_{\hat{1},\ldots,n-1} \cdots \overline{a}_{1,\ldots,\widehat{n-1}}) \vdash \mathrm{tp}(\overline{a}_{1,\ldots,n-1}/\overline{a}_{\hat{1},\ldots,n} \cdots \overline{a}_{1,\ldots,\widehat{n-1},n}).$$

Surprisingly the following are true for any *simple* theories.

Theorem 1.1. *(T is simple.) Let $n \geq 3$ and let T have $(n-1)$-uniqueness. Then n-uniqueness holds if and only if $B(n)$ holds.*

Theorem 1.2. *Suppose that T is a simple theory that has the $(n-1)$-uniqueness property and the $(n+1)$-existence property for some $n \geq 3$. Then T has $B(n)$.*

Corollary 1.13. *T simple. Assume T has k-uniqueness for all $3 \leq k < n$ $(4 \leq n)$. Then the following are equivalent.*

(1) *T has n-uniqueness.*

(2) *T has $(n+1)$-amalgamation.*

(3) *$B(n)$ holds in T.*

If T is stable (so 2-uniqueness holds), then above holds for $n = 3$.

Theorem 1.14. *T simple. Assume T has $B(k)$ for all $3 \leq k < n$. Then T has $K(n)$-amalgamation iff T has n-amalgamation.*

Now we talk about non-closed amalgamation.

Definition 1.15. Fix an algebraically closed set A and let $\mathcal{C}_A = \mathcal{C}_A(T)$ be the category of sets containing A with A-elementary embeddings. Suppose that $S \subseteq \mathcal{P}(n)$ is closed under subsets and $f : S \to \mathcal{C}_A$ is a functor.

(1) For $1 \leq k < n$, a functor f is *k-skeletal (over A)* if
   (a) f is independent; and
   (b) For every $u \in S$, $f(u) = \bigcup\{\mathrm{acl}(f_u^v(v)) \mid v \subseteq u,\ |v| \leq k\}$.

(2) We say T has (n,k)-*amalgamation* over A if any k-skeletal functor $f : \mathcal{P}^-(n) \to \mathcal{C}_A$ can be extended to a k-skeletal functor $\hat{f} : \mathcal{P}(n) \to \mathcal{C}_A$. Recall that T has *n-amalgamation* (over A) if it has $(n, n-1)$-*amalgamation* (over A.)

Corollary 1.16. (1) *If T is $B(n-2)$-simple (i.e. simple having $B(3), ..., B(n-2)$) having n-amalgamation, then it has (n,k)-amalgamation for each $0 < k < n$. Conversely if T is $B(n-1)$-simple having $(n,1)$-amalgamation, then it has (n,k)-amalgamation for each $0 < k < n$.*

(2) *If T is $B(k)$-stable (i.e. stable having $B(3), ..., B(k)$), then $(n, k-1)$-amalgamation holds for $n \geq k$.*

Unlike stable theories, there are not much relationship between $B(3)$ and 4-amalgamation in general unstable simple theories.

The theory $T_{tet.free}$ of the tetrahedron free ternary graph is simple having $B(3)$ but by nature it does not have 4-amalgamation.

The following example, a variation of the previous example, does not have $B(3)$ but it has 4-amalgamation.

Example 1.17. Consider the class of models $(I, [I]^2, B;\ E, P;\ \mathrm{Pr}_1 : B \to [I]^2)$ where infinite I, $B = [I]^2 \times \{0,1\}$, the membership relation $E \subseteq I \times [I]^2$ are as before, and $P \subset [B]^3$ such that $\{(x,\delta),(y,\delta'),(w,\delta'')\} \in P$ implies x, y, w are edges of a triangle, and $\{(x,\delta+1),(y,\delta'),(z,\delta'')\} \notin P$.

Now let T be a model companion of the theory of the class.

Only under a certain associativity condition of a definable binary function (stable theories have this), 4-amalgamation implies $B(3)$.

2. Homology

We develop a certain homology theory related to (the failure of) n-amalgamation.

Definition 2.1. (1) For each $n < \omega$, let Δ_n be the set of increasing sequences of natural numbers of length $(n+1)$ and let $\Delta = \bigcup_{n<\omega} \Delta_n$.

(2) By a *face map* we mean a map the form

$$\partial_i^n = \partial_i : \langle s_0, \ldots, s_n \rangle \in \Delta_n \mapsto \langle s_0, \ldots, \hat{s}_i, \ldots, s_n \rangle \in \Delta_{n-1}$$

Now we use $\mathcal{C} = \mathcal{C}(T)$ to denote the category of all *small* subsets (not necessarily closed) in the monster model $\mathcal{M} = \mathcal{M}^{eq}$ of T with elementary embeddings.

Definition 2.2. An *n-simplex* is an i.p. functor $f : \mathcal{P}(s) \to \mathcal{C}$, where $s \in \Delta_n$. Namely, for $u \subseteq s$,

(1) $\{f_u^{\{i\}}(\{i\}) | i \in u\}$ is $f_u^{\emptyset}(\emptyset)$-independent;
(2) $f(u) \subseteq \operatorname{acl}(\bigcup\{f_u^{\{i\}}(\{i\}) |\ i \in u\})$.

The n-simplex f is said to be an *n-simplex over A* if for each $u \subseteq s$, we have $f_u^{\emptyset}(\emptyset) = A$; and is in addition said to be an *n-simplex of* $p \in S(A)$ if for each $i \in u$, we have $f_u^{\{i\}}(\{i\}) \models p$.

Denote the set of all n-simplices by $S_n\mathcal{C}$; the set $S\mathcal{C}$ is the union of $S_n\mathcal{C}$, $n < \omega$. Similarly use $S_n\mathcal{C}(A)$ and $S_n\mathcal{C}(p)$.

Definition 2.3. (1) The *face map* lifts naturally to $S\mathcal{C}$, so that

$$\partial_i = \partial_i^n : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$$

as

$$\partial_i f := f\lceil\mathcal{P}(\partial_i s),$$

where $\mathcal{P}(s) = \operatorname{dom} f$.

(2) A *k-face* of an n-simplex $f : \mathcal{P}(s) \to \mathcal{C}$ $(0 \leq k \leq n)$ is a k-simplex of the form $f \restriction \mathcal{P}(u)$, where u is some $(k+1)$-element subset of s. In particular, $\partial_i f$ is an $(n-1)$-face of f.

Definition 2.4. (1) We say an n-simplex f is *closed* if for any $u \in \operatorname{dom}(f)$,

$$f(u) = \operatorname{acl}(\bigcup\{f_u^{\{i\}}(\{i\}) |\ i \in u\}).$$

(2) A n-simplex is a *k-frame*, $0 \leq k \leq n$, if all its k-faces are closed and $f(v) = \bigcup\{f_v^u(u) |\ u \subseteq v \text{ and } |u| = k+1\})$ for all $v \in \operatorname{dom}(f)$ with $|v| \geq k+1$.

From now on unless obviously otherwise, all simplices are closed.

Definition 2.5. (1) The free abelian group generated by the n-simplices is denoted by $C_n\mathcal{C}$. An element $c \in C_n\mathcal{C}$ is called an *n-chain*. Hence $c = \sum_j k_j f_j$ where $k_j \in \mathbb{Z}$ and f_j an n-simplex.

(2) For an n-simplex f, define the boundary operator

$$\partial f = \partial^n f := \sum_{i=0}^{n} (-1)^i \partial_i^n f,$$

so that ∂f is an $(n-1)$-chain. Extend $\partial : C_n\mathcal{C} \to C_{n-1}\mathcal{C}$ as, given $c = \sum_j k_j f_j \in C_n\mathcal{C}$,

$$\partial c := \sum_j k_j \partial f_j = \sum_j \sum_{i=0}^{n} k_j (-1)^i \partial_i^n f_j.$$

As usual, one can check $\partial^{n-1}\partial^n c = 0$ for any n-chain c.

Definition 2.6. (1) We say an n-chain $c \in C_n\mathcal{C}$ is an *n-cycle* if $\partial c = 0$. Let $Z_n(T)$ be the set of all n-cycles in $C_n\mathcal{C}$. Namely $Z_n(T) = \ker \partial^n$. (Similarly we define $Z_n(A)$ and $Z_n(p)$.)
(2) We say an n-chain $c \in C_n\mathcal{C}$ is an *n-boundary* if $c = \partial d$ for some $(n+1)$-chain d. Let $B_n(T)$ be the set of all n-boundaries in $C_n\mathcal{C}$. Namely $B_n(T) = \mathrm{im}\ \partial^{n+1}$. (Similarly we define $B_n(T;A)$ and $B_n(p)$.)
(3) Let $H_n(T) = Z_n(T)/B_n(T)$ be the *nth homology group of T* Similarly define $H_n(A)$ or $H_n(p)$.

Definition 2.7. An *n-shell* is an n-chain of the form

$$\sum_{0 \le i \le n+1} (-1)^i f_i,$$

where $f_0, \ldots, f_{n+1}$ are n-simplices such that for $0 \le i < j \le n+1$, we have $\partial_i f_j = \partial_{j-1} f_i$.

For example, if f is any $(n+1)$-simplex, then ∂f is an n-shell.

Now we can rephrase n-amalgamation, $K(n)$-amalgamation as follows.

Definition 2.8. (1) A theory has *n-amalgamation* over A if for every $(n-2)$-shell $c \in C_{n-2}\mathcal{C}(A)$, there is an $(n-1)$-simplex $f \in S_{n-1}\mathcal{C}(A)$ such that $c = \partial f$.
(2) A theory has *$K(n)$-amalgamation* if for every $(n-2)$-shell $c \in C_{n-2}\mathcal{C}$ of 0-frame, there is an $(n-1)$-simplex $f \in S_{n-1}\mathcal{C}$ of 0-frame such that $c = \partial f$.

3. Computation of homology groups

Observation 3.1. Let $\{A_i |\ i < \bar{\kappa} = |\mathcal{M}|\}$ be the collection of all *small* closed sets. Then $H_n(T) = \oplus_{i<\bar{\kappa}} H_n(A_i)$:

Note that the chain group $C_n\mathcal{C}(T) = \oplus_i C_n\mathcal{C}(A_i)$. The boundary operator ∂ sends an n-chain to $(n-1)$-chain componentwise. Thus the observation follows.

It says, unless trivial $H_n(T)$ doesn't make much sense; or if the bases are distinct then the spaces are disconnected. There is another reason that it would be better to restrict our attention to $H_n(p)$ rather than $H_n(A)$.

Fix $p \in S(A)$ (suppress $A = \emptyset = \mathrm{acl}(\emptyset)$) with $a = \mathrm{acl}(a)$ for $a \models p$.

Lemma 3.2. $H_0(p) = \mathbb{Z}$.

Recall that having n-CA means having k-amalgamation for each $k \leq n$.

Lemma 3.3. *Suppose $n \geq 2$. Assume T has n-CA over A. Then every $(n-1)$-cycle is a sum of $(n-1)$-shells. Namely, for $c \in Z_{n-1}(p)$, there are finitely many $(n-1)$-shells $c_j \in Z_{n-1}(p)$ such that $c = \sum_j k_j c_j$ $(k_j \in \mathbb{Z})$.*

Corollary 3.4. *(Shell Lemma) Assume T has n-CA over A for $n \geq 2$. Then $\{[c] |\ c$ is an $(n-1)-$shell $\}$ generates $H_{n-1}(p)$. In particular, if any $(n-1)$-shell is a boundary then so is any $(n-1)$-cycle.*

Corollary 3.5. *Suppose for $n \geq 3$, T has n-CA over A. Then $H_{n-2}(p) = 0$.*

Corollary 3.6. *$H_1(p) = 0$, $H_1(T) = 0$ for any simple theory.*

Definition 3.7. Let $n \geq 1$. By an *n-fan* (of p) we mean an n-chain of the form

$$\pm \sum_{i \in \{0,..,\hat{k},...,n\}} (-1)^i f_i,$$

where $f_0, \ldots, f_n$ are n-simplices such that whenever $0 \leq i < j \leq n$, we have $\partial_i f_j = \partial_{j-1} f_i$. In other words, an *$n$-fan* is $\pm$(an n-shell without a term).

Lemma 3.8. *(Prism Lemma) Suppose that a 2-shell (of p) $c = f_{123} - f_{023} + f_{013} - f_{012}$; and a 2-fan $g_{567} - g_{467} + g_{457}$ are given, where $\mathrm{dom}(f_{ijk}) = \mathcal{P}(\{i,j,k\})$ and the same holds for $\mathrm{dom}(g_{ijk})$. Then there is a 2-simplex g_{456} and a 3-chain e such that*

$$d := g_{567} - g_{467} + g_{457} - g_{456}$$

forms a 2-shell; and

$$\partial e = c - d.$$

4. Examples

Recall the example M_0 not having 4-amalgamation over $\emptyset$.

Example 4.1. Consider $[I]^2 = \{\{a,b\} | a \neq b \in I\}$ where I infinite. Let $B = [I]^2 \times \{0,1\}$ where $\{0,1\} = \mathbb{Z}_2$.

Also let $E \subseteq I \times [I]^2$ be a membership relation, and let P be a subset of B^3 such that $((w_1,\delta_1)(w_2,\delta_2)(w_3,\delta_3)) \in P$ iff there are distinct $a_1, a_2, a_3 \in I$ such that for $\{i,j,k\} = \{1,2,3\}$, $w_i = \{a_j, a_k\}$ (i.e. w_1, w_2, w_3 are edges of a triangle), and $\delta_1 + \delta_2 + \delta_3 = 0$.

Let $M_0 = (I, [I]^2, B;\ E; P;\ \mathrm{Pr}_1 : B \to [I]^2)$. Then M_0 is stable.

Let $p \in S(\emptyset)$ be the type of an element of I.

Claim 4.2. $H_2(p) = \mathbb{Z}_{\not{2}}$.

Define an augmentation map

$$\epsilon : S_2\mathcal{C}(p) \to \mathbb{Z}_{\not{2}}$$

as follows: Let f be a 2-simplex with $\mathrm{dom}(f) = \mathcal{P}(\langle n_0, n_1, n_2 \rangle)$ ($n_0 < n_1 < n_2$). For $i < j \le 3$, there is $a_{ij} \in [I]^2$ such that $f(\langle n_i, n_j \rangle) = \mathrm{acl}((a_{ij}; (a_{ij}, 0), (a_{ij}, 1)))$. Then we let $\epsilon(f) = 0$ if and only if

$$P(f_*(a_{01}, 0), f_*(a_{12}, 0), f_*(a_{02}, 0))$$

holds.

The map ϵ obviously extends as an homomorphism $\epsilon : C_2\mathcal{C} \to \mathbb{Z}_{\not{2}}$. Note that a 2-shell c is the boundary of 3-simplex iff $\epsilon(c) = 0$. Thus for any 2-boundary c, we have $\epsilon(c) = 0$. Hence ϵ induces a homomorphism $\epsilon_* : H_2(p) \to \mathbb{Z}_{\not{2}}$. Note that there is a 2-shell d such that $\epsilon(d) = 1$. Hence ϵ_* is onto. Indeed $d+d$ is a 2-boundary (see next page). Hence to finish the proof of Claim, we need to show that $[d]$ is the only generator (of order 2). By Prism Lemma, we can show that for an arbitrary 2-shell c, either $[c] = [0]$ or $[c] = [d]$. Now the last task is to argue that this is true for *any* 2-cycle (not just a 2-shell). But Shell Lemma asserts this. We have verified Claim.

Corollary 4.3. *The map ϵ_* is an isomorphism. Thus for any 2-cycle c if $\epsilon_*(c) = 0$, then c is a 2-boundary.*

Now consider the tetrahedron free graph. It is simple not having 4-amalgamation. Let p be the unique 1-type over $\emptyset$. Now any 2-shell is the boundary of a 3-simplex or that of a 3-fan. Hence by Shell Lemma, we have $H_2(p) = 0$.

Finally in [4], it is shown that in a stable theory, any finite abelian group can occur as $H_2(p)$. Given any finite abelian group G, let T_G be the complete theory of a connected groupoid $\mathcal{G}$ with infinitely many objects such that for some (any) $a \in \mathrm{Ob}(G)$, $\mathrm{Mor}(a, a)$ is isomorphic to G, where the language is the usual language for categories (this is a two-sorted language with an object sort and a morphism sort, and basic function symbols for composition, the source and target of a morphism, and a function id mapping any object to its corresponding identity morphism). The theory is totally categorical (hence stable). Let p be the strong type over $\emptyset$ of the algebraic closure of some (any) object of the groupoid $\mathcal{G}$. Then $H_2(p) = G$.

Question 4.4. Is there an example of type p such that $H_2(p)$ is infinite?

References

[1] T. de Piro, B. Kim and J. Millar 'Constructiong the hyperdefinable group from the group configuration', *Journal of Math. Logic* 6 (2006) 121-139.

[2] J. Goodrick and A. Kolesnikov, 'Groupoids, covers, and 3-uniqueness in stable theories', to appear in *J. Symbolic Logic.*

[3] J. Goodrick, B. Kim, and A. Kolesnikov, 'Amalgamation functors and the boundary properties', in preparation.

[4] J. Goodrick, B. Kim, and A. Kolesnikov, 'A homology theory for complete types in a simple theory', in preparation.

[5] E. Hrushovski, 'Groupoids, imaginaries and internal covers', preprint. arXiv:math.LO/0603413

[6] B. Kim 'Recovering the hyperdefinable group action in the group configuration theorem', *Journal of Symbolic Logic* 75 (2010) 12-24.

[7] B. Kim, A. Kolesnikov and A. Tsuboi 'Generalized amalgamation and n-simplicity,' *Annals of Pure and Applied Logic* 155 (2008) 97-114.

[8] B. Kim and A. Pillay, 'Simple theories', *Ann. Pure and Applied Logic* 88 (1997) 149-164.

[9] A. Kolesnikov, 'n-simple theories', *Ann. Pure and Applied Logic* 131 (2005) 227-261.

Department of Mathematics, Yonsei University, 134 Shinchon-dong, Seodaemun-gu, Seoul 120-749, South Korea

E-mail address: bkim@yonsei.ac.kr